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Statistics of a Chi-Square Random Variable Obtained from Independent Gaussian Samples with a Non-Zero Mean and Arbitrary Variance

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Statistics of a chi-square random variable obtained from independent Gaussian samples with a non-zero mean and arbitrary variance

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ABSTRACT

The mean and variance of a chi-square random variable are generally given for the case in which the chi-square random variable is derived from a process having a zero mean and unit variance. In this report, the mean and variance of the random variable found by squaring and summing N samples of an independent Gaussian process with a non-zero mean and arbitrary variance is derived.

Statistics of a chi-square random variable obtained from independent Gaussian samples with a non-zero mean and arbitrary variance

1. Introduction

In this report, general expressions are derived for the mean and variance of a random variable that is found by squaring and summing N independent samples of a Gaussian process x_i, which has a non-zero mean and an arbitrary variance (i.e. $\sum_{i=1}^{N} x_i^2$).

A common structure which generates such a random variable is an energy detector (Figure 1). Each output sample is obtained by squaring and summing N input samples. This is the optimal detector for an unknown signal buried in white Gaussian noise, and is often used as a post-processor for other routines (for example, the output of a beamformer may be run through an energy detector to determine if a signal is present).

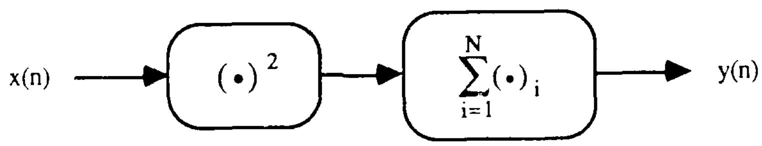

Figure 1. Energy detector.
Finds the energy in N samples of x(n). Each output sample is produced by squaring and summing N input samples.

Chi-square random variables

Let $z_1, z_2, z_3, \ldots$ be normally distributed, independent random variables with zero mean and a variance of one, and define a new random variable

$$\chi_N^2 = z_1^2 + z_2^2 + z_3^2 + \ldots + z_N^2 \tag{1}$$

The variable in (1) is called a chi-square random variable with N degrees of freedom. The density function of χ_N^2 approaches that of a normally distributed random variable for large N ($N > 30$), and is non-symmetric for smaller N. The mean and variance of the chi-square random variable in (1) is given by [1, page 105].

$$\mu_{\chi_N^2} = E\left\{\chi_N^2\right\} = N \tag{2}$$

$$\sigma_{\chi_N^2}^2 = var\left[\chi_N^2\right] = 2N \tag{3}$$

Details about the density and distribution functions may be found in [1, Section 4.2.2]. Random variables with a non-zero mean that are squared and summed have a non-central chi-square distribution [3,4].

A time series x can always be transformed to have a mean of zero and variance equal to one by defining the standardized variable z_i to be

$$z_i = \frac{x_i - \mu}{\sigma} \tag{4}$$

In some cases, it is desirable to find the statistics of the random variable formed by squaring and summing N values of x, which does not necessarily have a zero mean or variance equal to one. In the next section, general expressions for the mean and variance of a random variable that is obtained from squaring and summing independent Gaussian samples with a non-zero mean μ and arbitrary variance σ^2 are derived.

2. Derivation

Given the standardized random variable in (4), a chi-square random variable with N degrees of freedom is obtained by squaring the z_i and summing over N samples.

$$\chi_N^2 = \sum_{i=1}^{N} z_i^2 \tag{5}$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=1}^{N}(x_i - \mu)^2\right]$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=1}^{N}(x_i^2 - 2\mu x_i + \mu^2)\right]$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=1}^{N} x_i^2 - 2\mu\sum_{i=1}^{N} x_i + N\mu^2\right] \tag{6}$$

$$E\left\{\chi_N^2\right\} = \frac{1}{\sigma^2} E\left\{\sum_{i=1}^{N} x_i^2\right\} - \frac{2\mu}{\sigma^2} E\left\{\sum_{i=1}^{N} x_i\right\} + \frac{N\mu^2}{\sigma^2}$$

$$= \frac{1}{\sigma^2} E\left\{\sum_{i=1}^{N} x_i^2\right\} - \frac{N\mu^2}{\sigma^2}$$

Solving for the N squared and summed samples of x_i, and using Equation (2) gives

$$E\left\{\sum_{i=1}^{N} x_i^2\right\} = N\sigma^2 + N\mu^2$$

$$= N(\sigma^2 + \mu^2) \tag{7}$$

By definition, the variance of the chi-square random variable is given by

$$var\left[\chi_N^2\right] = E\left\{\left[\chi_N^2 - \mu_{\chi_N^2}\right]\left[\chi_N^2 - \mu_{\chi_N^2}\right]\right\} \tag{8}$$

Using (2) and (6), and multiplying both sides by σ^4 gives

$$\sigma^4\ var\left[\chi_N^2\right] = E\left\{\left[\sum_{i=1}^{N} x_i^2 - 2\mu\sum_{i=1}^{N} x_i + N\mu^2 - N\sigma^2\right]\left[\sum_{j=1}^{N} x_j^2 - 2\mu\sum_{j=1}^{N} x_j + N\mu^2 - N\sigma^2\right]\right\}$$

Replacing the expression on the left side with (3) and expanding yields

$$\begin{aligned} 2N\sigma^4 = E\Big\{&\Big(\sum_{i=1}^{N} x_i^2\Big)^2 - 2\mu\sum_{i=1}^{N} x_i^2\sum_{j=1}^{N} x_j + N\mu^2\sum_{i=1}^{N} x_i^2 - N\sigma^2\sum_{i=1}^{N} x_i^2 - 2\mu\sum_{i=1}^{N} x_i^2\sum_{j=1}^{N} x_j \\ &+ 4\mu^2\Big(\sum_{i=1}^{N} x_i\Big)^2 - 2N\mu^3\sum_{i=1}^{N} x_i + 2N\mu\sigma^2\sum_{i=1}^{N} x_i + N\mu^2\sum_{i=1}^{N} x_i^2 - 2N\mu^3\sum_{i=1}^{N} x_i \\ &+ N^2\mu^4 - N^2\mu^2\sigma^2 - N\sigma^2\sum_{i=1}^{N} x_i^2 + 2N\mu\sigma^2\sum_{i=1}^{N} x_i - N^2\mu^2\sigma^2 + N^2\sigma^4\Big\} \end{aligned}$$

$$\begin{aligned} = E\Big\{&\Big(\sum_{i=1}^{N} x_i^2\Big)^2 - 4\mu\sum_{i=1}^{N} x_i^2\sum_{j=1}^{N} x_j + 2N\mu^2\sum_{i=1}^{N} x_i^2 - 2N\sigma^2\sum_{i=1}^{N} x_i^2 - 4N\mu^3\sum_{i=1}^{N} x_i \\ &+ 4N\mu\sigma^2\sum_{i=1}^{N} x_i - 2N^2\mu^2\sigma^2 + N^2\mu^4 + N^2\sigma^4 + 4\mu^2\Big(\sum_{i=1}^{N} x_i\Big)^2\Big\} \end{aligned}$$

Using Equations (A1) - (A4) from the Appendix in the above equation results in

$$
\begin{aligned}
2N\sigma^4 \;=\;& E\left\{\left(\sum_{i=1}^{N} x_i^2\right)^2\right\} - 4\mu\left(2N\mu\sigma^2 + N^2\mu\sigma^2 + N^2\mu^3\right) + 2N^2\mu^2\left(\sigma^2+\mu^2\right) \\
& - 2N^2\sigma^2\left(\sigma^2+\mu^2\right) - 4N^2\mu^4 + 4N^2\mu^2\sigma^2 - 2N^2\mu^2\sigma^2 + N^2\mu^4 \\
& + N^2\sigma^4 + 4\mu^2\left(N\sigma^2 + N^2\mu^2\right) \\
=\;& E\left\{\left(\sum_{i=1}^{N} x_i^2\right)^2\right\} - 8N\mu^2\sigma^2 - 4N^2\mu^2\sigma^2 - 4N^2\mu^4 + 2N^2\mu^2\sigma^2 + 2N^2\mu^4 \\
& - 2N^2\sigma^4 - 2N^2\mu^2\sigma^2 - 4N^2\mu^4 + 4N^2\mu^2\sigma^2 + N^2\mu^4 - 2N^2\mu^2\sigma^2 + N^2\sigma^4 \\
& + 4N\mu^2\sigma^2 + 4N^2\mu^4 \\
=\;& E\left\{\left(\sum_{i=1}^{N} x_i^2\right)^2\right\} - 4N\mu^2\sigma^2 - 2N^2\mu^2\sigma^2 - N^2\mu^4 - N^2\sigma^4
\end{aligned}
$$

Therefore,

$$
E\left\{\left(\sum_{i=1}^{N} x_i^2\right)^2\right\} = 2N\sigma^4 + 4N\mu^2\sigma^2 + 2N^2\mu^2\sigma^2 + N^2\mu^4 + N^2\sigma^4 \tag{9}
$$

For a random variable y,

$$
var\{y\} = E\{y^2\} - \left(E\{y\}\right)^2
$$

therefore the variance of $\sum_{i=1}^{N} x_i^2$ is given by

$$
var\left[\sum_{i=1}^{N} x_i^2\right] = E\left\{\left(\sum_{i=1}^{N} x_i^2\right)^2\right\} - \left(E\left\{\sum_{i=1}^{N} x_i^2\right\}\right)^2
$$

Using Equations (9) and (7) results in

$$
\begin{aligned}
var\left[\sum_{i=1}^{N} x_i^2\right] =\;& E\left\{\left(\sum_{i=1}^{N} x_i^2\right)^2\right\} - N^2\left(\sigma^2+\mu^2\right)^2 \\
=\;& 2N\sigma^4 + 4N\mu^2\sigma^2 + 2N^2\mu^2\sigma^2 + N^2\mu^4 + N^2\sigma^4 - N^2\sigma^4 - 2N^2\mu^2\sigma^2 - N^2\mu^4
\end{aligned}
$$

$$= 2N\sigma^4 + 4N\mu^2\sigma^2 \tag{10}$$

3. Summary

If x is a Gaussian random process and has mean μ and variance σ^2, then

$$E\left\{\sum_{i=1}^{N} x_i^2\right\} = N(\sigma^2 + \mu^2)$$

$$var\left[\sum_{i=1}^{N} x_i^2\right] = 2N\sigma^4 + 4N\mu^2\sigma^2$$

Note that when $\mu=0$ and $\sigma^2=1$ these equations reduce to (2) and (3).

Appendix: Expected values of various summations

1. $$E\left\{\sum_{i=1}^{N} x_i\right\} = N\mu \tag{A1}$$

2. $$E\left\{\sum_{i=1}^{N} x_i^2\right\} = N(\sigma^2 + \mu^2) \tag{A2}$$

This is Equation (7) and was derived in the main text.

3. $$E\left\{\left(\sum_{i=1}^{N} x_i\right)^2\right\} = N\sigma^2 + N^2\mu^2 \tag{A3}$$

Derivation

Let x_i be normally distributed random variables and define $\tilde{x} = \sum_{i=1}^{N} x_i$, then $\tilde{x}$ is a normal random variable with mean $N\mu$ and variance $N\sigma^2$. The variance of $\tilde{x}$ may be written as

$$var\left[\tilde{x}\right] = E\left\{\tilde{x}^2\right\} - \left(E\{\tilde{x}\}\right)^2$$

so that

$$\begin{aligned} E\left\{\tilde{x}^2\right\} &= E\left\{\left(\sum_{i=1}^{N} x_i\right)^2\right\} \\ &= var(\tilde{x}) + \left(E\{\tilde{x}\}\right)^2 \\ &= N\sigma^2 + N^2\mu^2 \end{aligned}$$

4. $$E\left\{\sum_{i=1}^{N} x_i^2 \sum_{j=1}^{N} x_j\right\} = 2N\mu\sigma^2 + N^2\mu\sigma^2 + N^2\mu^3 \tag{A4}$$

Derivation

$$E\left\{\sum_{i=1}^{N} x_i^2 \sum_{j=1}^{N} x_j\right\} = E\left\{\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^2 x_j\right\} = \sum_{i=1}^{N}\sum_{j=1}^{N} E\left\{x_i^2 x_j\right\}$$

When $i = j$ this becomes the single sum $\sum_{i=1}^{N} E\{x_i^3\}$. From [2, page 162],

$$E\{x_i^3\} = 3\mu\sigma^2 + \mu^3$$

so

$$\sum_{i=1}^{N} E\{x_i^3\} = N(3\mu\sigma^2 + \mu^3) \qquad i = j$$

When $i \neq j$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} E\{x_i^2 x_j\} = \sum_{i=1}^{N}\sum_{j=1}^{N} E\{x_i^2\}\, E\{x_i\}$$

since the processes are independent. Using $E\{x_i\} = \mu$, and

$$E\{x_i^2\} = var\{x\} + \mu^2 = \sigma^2 + \mu^2$$

results in

$$\sum_{i=1}^{N}\sum_{j=1}^{N} E\{x_i^2\}\, E\{x_i\} = \sum_{i=1}^{N}\sum_{j=1}^{N} \mu(\sigma^2 + \mu^2)$$

These are summed over all i and j *except* for the case $i = j$, so there are $N(N-1)$ of them, giving

$$\sum_{i=1}^{N}\sum_{j=1}^{N} E\{x_i^2\}\, E\{x_i\} = N(N-1)\,\mu\,(\sigma^2 + \mu^2) \qquad i \neq j$$

Adding the cases for $i = j$ and $i \neq j$ together gives

$$E\left\{\sum_{i=1}^{N} x_i^2 \sum_{j=1}^{N} x_j\right\} = N(3\mu\sigma^2 + \mu^3) + N(N - 1)\mu(\sigma^2 + \mu^2)$$

$$= 2N\mu\sigma^2 + N^2\mu\sigma^2 + N^2\mu^3$$

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